

$SO_q(N)$ -isotropic Harmonic Oscillator on the Quantum Euclidean Space \mathbb{R}_q^N ¹

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Abstract

We briefly describe the construction of a consistent q -deformation of the quantum mechanical isotropic harmonic oscillator on ordinary \mathbb{R}^N space.

Introduction

In this report we briefly describe how to formulate a consistent quantum-mechanical one-particle system on a “noncommutative manifold” (the quantum Euclidean space \mathbb{R}_q^N) with a “non-classical” space symmetry (the one carried by the quantum group $SO_q(N)$). This system is a q -deformation of the isotropic harmonic oscillator on ordinary \mathbb{R}^N -space; $q \rightarrow 1$ is the corresponding “classical limit”. The guiding idea is to mimic in a q -deformed setting the ordinary realization of one-particle quantum mechanics in configuration space. Essential references are [2, 4, 5, 6, 7], to which we refer the reader for details and proofs.

Incidentally, the problem considered here is rather different from that of the q -deformed harmonic oscillators treated by other authors [1]: there, creation/annihilation operators with some prescribed commutations relation are postulated from the very beginning without any reference to a geometrical framework. The deformation considered there concerned the well-known hidden $su(n)$ symmetries of the harmonic

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oscillator hamitonians. Here, on the contrary, the deformation concerns the rotation symmetry of the space itself and creation/destruction operators are constructed out of the deformed “ coordinates ” and “ derivatives ”.

The general context for a proper location of the work is a research program [14, 10] consisting of: 1) a continuous deformation of both space(time) geometry and its related fundamental symmetries through socalled quantum (and/or braided) spaces/groups (which are characteristic examples of noncommutative geometries); 2) the formulation of quantum mechanics and quantum field theory on them. The main physical motivation for such a program is that it represents a radical approach to long-standing problems in QFT, e.g. ultraviolet divergencies, quantizing gravity, etc, since one modifies the microscopic structure of spacetime. Of course the effects of such a modification should be practically undetectable at least in the domain of observability of already well-explained physical phenomena.

The approach is an algebraic one which seems in deep agreement with the spirit of quantum mechanics. In fact, one of the essential features of the latter is that it invalidates the classical geometrical description of the state of a physical system as a point in the corresponding phase space; the naive formulation of this fact are Heisenberg’s indetermination relations. The notion of “ points ” in continuous phase space is replaced by the more general notion of a (\mathbf{C}^*) -algebra of operators acting on a Hilbert space; consequently, the geometry of the phase space is broken. Geometry may be recovered as a useful structure for the spectral decomposition of operators. For instance, the spectral decomposition of the vector components of the position operator of one quantum particle typically is realized on a manifold coinciding with classical configuration space. This is due to the axiom that these components commute, $[x^i, x^j] = 0$. In a noncommutative-geometric approach to quantum mechanics one essentially releases the latter axiom. One would be interested in a tiny deformation of these commutation relations, but of course this can be done in infinitely too many ways.

Requiring that the deformed relations keep track of the spacetime symmetries puts severe constraints to the possible deformations. Quantum spaces and related quantum groups have been conceived to satisfy them, in that they are *coupled* deformations of space(time) geometries and their fundamental symmetries.

1 Preliminaries

$\hat{R}_q := ||\hat{R}_{hk}^{ij}||$ will denote the braid matrix of the quantum group $SO_q(N)$ [3], $C := ||C_{ij}||$ the corresponding q-deformed metric matrix. Here i, j, h, k belong to $\{-n, -n+1, \dots, -1, 0, 1, \dots, n\}$ if $N = 2n+1$, and to $\{-n, -n+1, \dots, -1, 1, \dots, n\}$ if $N = 2n$. Indices are raised and lowered through the metric matrix C , e.g. $a_i = C_{ij}a^j$, $a^i = C^{ij}a_j$; C is not symmetric an coincides with its inverse, $C^{ij} = C_{ij}$.

Both C and \hat{R} depend on q and are real for $q \in \mathbf{R}$; explicit expressions can be

found in Ref. [3]. \hat{R} admits the very useful decomposition

$$\hat{R}_q = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{1-N}\mathcal{P}_1. \quad (1.1)$$

$\mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1$ are the projection operators onto the three eigenspaces of \hat{R} (the latter have respectively dimensions $\frac{N(N+1)}{2} - 1, \frac{N(N-1)}{2}, 1$): they project the tensor product $x \otimes x$ of the fundamental corepresentation x of $SO_q(N)$ into the corresponding irreducible corepresentations (the symmetric modulo trace, antisymmetric and trace, namely the q -deformed versions of the corresponding ones of $SO(N)$). The projector \mathcal{P}_1 is related to the metric matrix C by $\mathcal{P}_1^{ij}_{hk} = \frac{C^{ij}C_{hk}}{Q_N}$; the factor Q_N is defined by $Q_N := C^{ij}C_{ij}$, and $Q_N \rightarrow N$. $\hat{R}^{\pm 1}, C$ satisfy the relations

$$[f(\hat{R}), P \cdot (C \otimes C)] = 0 \quad f(\hat{R}_{12})\hat{R}_{23}^{\pm 1}\hat{R}_{12}^{\pm 1} = \hat{R}_{23}^{\pm 1}\hat{R}_{12}^{\pm 1}f(\hat{R}_{23}) \quad (1.2)$$

(P is the permutator: $P_{hk}^{ij} := \delta_k^i \delta_h^j$ and f is any polynomial function); in particular this holds for $f(\hat{R}) = \hat{R}^{\pm 1}, \mathcal{P}_A, \mathcal{P}_S, \mathcal{P}_1$.

The unital algebra $Diff(\mathbf{R}_q^N)$ of differential operators on the real quantum euclidean space \mathbf{R}_q^N is defined as the space of formal series in the (ordered) powers of the $\{x^i\}, \{\partial_i\}$ variables (the q -deformed coordinates and derivatives, respectively) with complex coefficients, modulo the commutation relations

$$\mathcal{P}_A^{ij}_{hk} x^h x^k = 0. \quad \mathcal{P}_A^{ij}_{hk} \partial^h \partial^k = 0. \quad (1.3)$$

and the derivation relations

$$\partial_i x^j = \delta_i^j + q \hat{R}_{ik}^{jh} x^k \partial_h. \quad (1.4)$$

The subalgebra $Fun(\mathbf{R}_q^N)$ of “ functions ” on \mathbf{R}_q^N is generated by $\{x^i\}$ only. We only give as an example relations (1.3) in the case $N=3$. They amount to the three independent relations

$$x^{-1}x^0 - qx^0x^{-1} = 0 \quad x^0x^1 - qx^1x^0 = 0 \quad x^1x^{-1} - x^{-1}x^1 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x^0x^0; \quad (1.5)$$

in the limit $q \rightarrow 1$ we get back commuting coordinates and therefore ordinary geometry. More details can be found in Ref. [2] and in the contribution of U. Carow-Watamura & S. Watamura to these proceedings.

If $q \in \mathbf{R}^+$ one can introduce an antilinear involutive antihomomorphism $*$:

$$*^2 = id \quad (AB)^* = B^*A^* \quad (1.6)$$

on $Diff(\mathbf{R}_q^N)$ (the “complex conjugation”). On the basic variables x^i $*$ is defined by

$$(x^i)^* = x^j C_{ji} \quad (1.7)$$

whereas the complex conjugates of the derivatives ∂^i are not combinations of the derivatives themselves. It is useful to introduce barred derivatives $\bar{\partial}^i$ through

$$(\partial^i)^* = -q^{-N} \bar{\partial}^i C_{ji}. \quad (1.8)$$

They satisfy relation (1.3)₂ and the analog of (1.4) with q, \hat{R} replaced by q^{-1}, \hat{R}^{-1} . These $\bar{\partial}$ derivatives can be expressed as nonlinear functions of x, ∂ [11].

By definition a $Fun(SO_q(N))$ -scalar $I(x, \partial) \in Diff(\mathbf{R}_q^N)$ transforms trivially under the coaction associated to the quantum group of symmetry $SO_q(N, \mathbf{R})$ [3]. Any q -scalar polynomial $I(x, \partial) \in Diff(\mathbf{R}_q^N)$ of degree $2p$ in x, ∂ can be expressed as an ordered polynomial in two particular q -scalar variables (see for instance Appendix C of [12]), namely the square lenght $x C x := x^i C_{ij} x^j$ and the laplacian $\Delta := \partial^i C_{ij} \partial^j$. In Ref. [6] the Hopf algebra $U_q(so(N))$ [3] was realized as the subalgebra of $Diff(\mathbb{R}_q^N)$ characterized by the condition that its elements commute with the $Fun(SO_q(N))$ -scalar elements of $Diff(\mathbb{R}_q^N)$. One set of generators of such a subalgebra is $\{l^{ij}\}_{i \neq j}$, where

$$l^{ij} := \mathcal{P}_A^{ij} x^h \partial^k \Lambda^{-1} \quad (1.9)$$

and Λ is the “dilaton” defined by $\Lambda x^i = q x^i \Lambda$, $\Lambda \partial^i = q^{-1} \partial^i \Lambda$. Each l^{ij} can be interpreted as a q -deformed angular momentum component, since it commutes with q -scalars and in the classical limit reduces to the ordinary angular momentum component generating a rotation in the (i, j) plane of \mathbb{R}^N , $l^{ij} \xrightarrow{q \rightarrow 1} x^i \partial^j - x^j \partial^i$. $l \cdot l := l^{ij} l_{ji}$ itself is a scalar and therefore commutes with each l^{hk} ; it is a quadratic function of the quadratic casimir of $U_q(so(N))$ and will be called the “square angular momentum”. A linear function B of the quadratic casimir is

$$B := \Lambda^{-1} (1 + \frac{q^2 - 1}{1 + q^{2-N}} x^i \partial_i); \quad \Rightarrow \quad 1 = (B)^2 - \frac{(q^2 - 1)(q^2 - q^{-2})}{(1 + q^{2-N})(1 + q^{N-4})} (l \cdot l). \quad (1.10)$$

Incidentally, one can extend this realization of $U_q(so(N))$ by adding q -derivatives as generators of q -translations, in such a way to represent a Euclidean Hopf algebra $U_q(e^N)$ [8].

Riemann integration $\int dV$ over the Euclidean space \mathbb{R}^N is covariant under the action of the Euclidean group $E^N := \mathbb{R}^N \rtimes SO(N)$, and in particular is invariant under finite translations. In infinitesimal form, the latter invariance implies the validity of Stoke’s theorem for all integrable functions which are differentiable. E^N -covariance of the Riemann integration is essential in allowing a E^N -covariant description of ordinary physics. Similarly, in view of the formulation of E_q^N -covariant Physics, in Ref. [5, 7] a q -deformed integration $\int d_q V$ on \mathbb{R}_q^N was constructed by requiring its covariance w.r.t. the quantum Euclidean group $E_q^N := \mathbb{R}_q^N \rtimes SO_q(N)$ [9, 12]. Actually, its invariance w.r.t. to finite q -translations (or equivalently the validity of q -Stoke’s theorem, in infinitesimal form), together with the obvious requirement of linearity, are enough to allow its *construction*. Its $SO_q(N, \mathbf{R})$ -covariance follows from the $SO_q(N, \mathbf{R})$ -covariance of the “braided coaddition” [9] (or, equivalently, of the differential calculus [2] on \mathbb{R}_q^N).

In the classical case, if $f = P_n(x)\exp[-a|x|^2]$ (P_n denotes a polynomial of degree n in x and $|x|^2$ the square lenght), then

$$\partial^i P_n(x)\exp[-a|x|^2] = P_{n-1}(x)\exp[-a|x|^2] + P_{n+1}\exp[-a|x|^2]; \quad (1.11)$$

Stoke's theorem then implies

$$\int dV P_{n-1}(x)\exp[-a|x|^2] + \int dV P_{n+1}\exp[-a|x|^2] = 0. \quad (1.12)$$

This relation allows to recursively define the integral $\int dV f$ (for any function f of the same kind) in terms of $\int dV \exp[-a|x|^2]$ (which fixes the normalization of the integration); moreover, since the space of functions $\{P(x)\exp[-a|x|^2]\}$ is dense in $\mathcal{L}^1(\mathbb{R}^N)$, $\mathcal{L}^2(\mathbb{R}^N)$, one can approximate as much as desired integrals of other functions in this way. This situation can be summarized by saying that $\exp[-a|x|^2]$ can be taken as a “reference function” for the algebraic construction of Riemann integration. In an analogous way, $\int d_q V$ was algebraically *constructed* by choosing some q -deformed reference function (the simplest being the q -gaussian $\exp_q[-a|x|^2]$) and by imposing validity of the q -deformed Stoke's theorem

$$\int d_q V \partial^i f(x) = 0, \quad f \in Fun(\mathbb{R}_q^N). \quad (1.13)$$

q -integration, as Riemann one, also satisfies some other important properties, namely the reality condition

$$(\int d_q V f)^* = \int d_q V f^* \quad (1.14)$$

for any $q \in \mathbb{R}^+$, and the positivity condition

$$\int d_q V f^* f \geq 0, \quad \int d_q V f^* f = 0 \Leftrightarrow f = 0; \quad (1.15)$$

finally, it reduces to Riemann integration in the limit $q \rightarrow 1$, since the abovementioned algebraic recursion reduce to the classical one in the same limit.

2 Time-independent quantum mechanics on \mathbb{R}_q^N : the q -isotropic harmonic oscillator

The formal tools briefly presented in the preceding section are the “bricks” which we use to formulate time-independent quantum mechanics of one-particle systems on \mathbb{R}_q^N configuration space. The idea is to formulate the time-independent Schroedinger equation as a q -differential equation by introducing a q -deformed hamiltonian $h \in Diff(\mathbb{R}_q^N)$ such as

$$h = -\Delta + V(x) \quad (2.1)$$

and writing it as an eigenvalue equation

$$h\psi = E\psi \quad \psi \in Fun(\mathbb{R}_q^N). \quad (2.2)$$

A serious problem arises when requiring the hamiltonian to be a hermitean operator, since the above laplacian is not real. We solved it in a *nonstandard* way in Refs. [5, 7, 8] in two concrete models by postulating that each abstract vector $|u\rangle$ of the Hilbert space \mathcal{H} of states can be realized in two different ways $(\psi_u, \bar{\psi}_u)$ as a q -deformed wave-function (with a linear bijective map $\psi_u \leftrightarrow \bar{\psi}_u$), and correspondingly each abstract operator \mathcal{B} on \mathcal{H} in two different ways (b, \bar{b}) as a differential operator. b acts on ψ_u and \bar{b} on $\bar{\psi}_u$, in other words $\mathcal{B}|u\rangle := (b\psi_u, \bar{b}\bar{\psi}_u)$; ψ_u, b (resp. $\bar{\psi}_u, \bar{b}$) will be called the unbarred (resp. barred) realization of $|u\rangle, \mathcal{B}$. Of course this is done in such a way that all the physics (eigenvalues of observables, etc.) are the same in either realization. The crucial point of this approach is the definition of the scalar product of two vectors $|v\rangle, |u\rangle \in \mathcal{H}$ as the sum of the two “conjugate” terms:

$$\langle u|v\rangle := \int d_q V \bar{\psi}_u^* \psi_v + \int d_q V \psi_u^* \bar{\psi}_v. \quad (2.3)$$

Indeed $\langle \quad | \quad \rangle$ is manifestly sesquilinear and

$$\langle v|u\rangle^* = \left(\int d_q V \bar{\psi}_v^* \psi_u \right)^* + \left(\int d_q V \psi_v^* \bar{\psi}_u \right)^* \stackrel{(1.14)}{=} \int d_q V \psi_u^* \bar{\psi}_v + \int d_q V \bar{\psi}_u^* \psi_v = \langle u|v\rangle. \quad (2.4)$$

Its positivity has to be shown for each specific choice of h in formula (2.1); for the proof in the case of the harmonic oscillator see Ref. [5, 7]. The scalar product (2.3) is such that it allows to define a formally hermitean kinetic part pCp of the abstract hamiltonian $H = pCp + V$ by realizing it as the *pair* of conjugated laplacians $(-q^N \Delta, -q^{-N} \bar{\Delta})$ (here $\bar{\Delta} := \bar{\partial} C \partial$). In fact, it is easy to check that the hermitean conjugate \mathcal{B}^\dagger of an abstract operator $\mathcal{B} := (b, \bar{b})$ w.r.t. the scalar product (2.3) is given by the rule

$$\mathcal{B}^\dagger = (\bar{b}^*, b^*), \quad (2.5)$$

and consequently $(pCp)^\dagger = pCp$, since $\Delta^* = q^{-2N} \bar{\Delta}$.

Now we sketch how the previous program can be successfully developed in the case of the $Fun(SO_q(N, \mathbb{R}))$ -isotropic harmonic oscillator. We build $\forall q \in \mathbb{R}^+$ a sensible quantum mechanical model \mathbb{R}_q^N ($N \geq 3$) as the simplest q -deformation of the (time-independent) classical isotropic harmonic oscillator on ordinary \mathbb{R}^N ; correspondingly, the symmetry group $SO(N, \mathbb{R})$ of rotations of the hamiltonian is deformed into the quantum group symmetry $SO_q(N, \mathbb{R})$. The hamiltonian has a lower bounded energy spectrum and the scalar product is strictly positive for any $q \in \mathbb{R}^+$. Generalizing the classical algebraic construction, the Hilbert space of physical states is built applying creation operators to the (unique) ground state. Observables will be defined as hermitean operators, as usual. In particular we construct the observables hamiltonian, square angular momentum, angular momentum components, position

operators, momentum operators. As in the classical case, the first two and any angular momentum component will commute with each other; when $N = 3, 4$ they make up a complete set of commuting observables.

As a first task we have to fix within the differential algebra $Diff(\mathbb{R}_q^N)$ a suitable q -analogue h_ω of the classical hamiltonian $h_\omega^{cl} := -\Delta + \omega^2 x C x$ (x, ∂ being classical coordinates and derivatives) with characteristic constant ω . *A priori* we don't require it to be necessarily of the form (2.1). Minimal requirements on $h_\omega \in Diff(\mathbb{R}_q^N)$ are of course that:

- 1) it should be a $Fun(SO_q(N))$ -scalar (this is the meaning of the word “isotropic” in the q -deformed setting);
- 2) it should have a homogenous natural dimension $d(h_\omega) = d(\partial^2) = 2$ and should reduce to h_ω^{cl} in the limit $q \rightarrow 1$;
- 3) it should be the “unbarred” configuration-space realization of a hermitean operator H_ω on some Hilbert space \mathcal{H} (to be defined);
- 4) the spectrum of H_ω in \mathcal{H} should be bounded from below, in order that H_ω can be considered as the hamiltonian of a sensible (i.e. stable) quantum mechanical system.

Convenience suggests two further requirements. h_ω being a scalar, it commutes with $U_q(so(N))$ and in particular with the square angular momentum $l \cdot l := l^{ij} l_{ji}$; this means that when realized as operators, $h_\omega, l \cdot l$ can be diagonalized simultaneously. Requirements 1) - 4) still leave a great freedom in defining h_ω , which can be essentially summarized as its yet undefined $l \cdot l$ -dependence. By requiring that, as it happens for the classical isotropic harmonic oscillator,

- 5) Energy levels are $(l \cdot l)$ -independent

we essentially impose a trivial dependence of h_ω on $l \cdot l$. Moding out an essential dilaton-dependence and with a careful choice of the numerical coefficients, the final unbarred realization of the hamiltonian turns out to be

$$h_\omega := (-q^N \Delta + \omega^2 x C x). \quad (2.6)$$

We introduce the barred hamiltonian $\bar{h}_\omega := (-q^{-N} \bar{\Delta} + \omega^2 x C x)$ in such a way that $h_\omega^* = \bar{h}_\omega$ and we can build a formally hermitean abstract hamiltonian H_ω :

$$H_\omega := (h_\omega, \bar{h}_\omega) \quad \Rightarrow \quad H_\omega^\dagger = H_\omega. \quad (2.7)$$

Finally, we add the important requirement

- 6) we would like to algebraically solve the Schroedinger equation through the introduction of $Fun(SO_q(N))$ -vectors of energy creators $\vec{A}^+ := (A^{i+}) \in Diff(\mathbb{R}_q^N)$ and annihilators $\vec{A}^- := (A^{i-}) \in Diff(\mathbb{R}_q^N)$, as in the classical case. More precisely, we require commutations relations of the type

$$H_\omega A^{i\pm} = A^{i\pm} f^{i\pm}(H_\omega), \quad f^{i\pm}(t) \in \mathbb{C}[t]. \quad (2.8)$$

In fact, if relations of the form (2.8) are satisfied, given an eigenvector $|u\rangle$ of H_ω , $H_\omega|u\rangle = E|u\rangle$, then $A^{i\pm}|u\rangle$ would be an eigenvector of H_ω with eigenvalue $f^{i\pm}(E)$.

Proposition 1 *A solution of the above problem (2.8) is given by*

$$A^{i\pm} = (a^{i\pm}, \bar{a}^{i\pm}), \quad \begin{cases} a^{i\pm} = \Lambda^{-\frac{1}{2}}[x^i \beta^\pm(q, h_\omega) + \partial^i \gamma^\pm(q, h_\omega)] \\ \bar{a}^{i\pm} = \Lambda^{\frac{1}{2}}[x^i \beta^\pm(q^{-1}, h_\omega) + \partial^i \gamma^\pm(q^{-1}, h_\omega)] \end{cases} \quad (2.9)$$

³ where β^\pm, γ^\pm satisfy the condition

$$\frac{\beta^\pm(q, h_\omega)}{\gamma^\pm(q, h_\omega)} = \frac{q^{-N}}{1 + q^{2-N}}(h_\omega - qf^\pm). \quad (2.10)$$

and $f^{i\pm}$ coincide with one of the two solutions $f^\pm(h_\omega)$ of the equation

$$(qh_\omega - f)(q^{-1}h_\omega - f) = (1 + q^{2-N})^2 q^{N-2} \omega^2. \quad (2.11)$$

It is easy to check that in the limit $q \rightarrow 1$ both a^{i+} and \bar{a}^{i+} (respectively a^{i-} and \bar{a}^{i-}) go to the classical creation (resp. annihilation) operators of the ordinary isotropic harmonic oscillator.

Now let $\{|u\rangle, E\}$ denote the pair consisting of a eigenvector $|u\rangle = (\psi_u, \bar{\psi}_u)$ of H_ω and the corresponding eigenvalue E . We can now generate a “shower” of such pairs by means of iterated use of relations (2.8), (2.11):

$$\begin{array}{rcccl} \{ |u\rangle, E \} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \begin{array}{l} \{ A^{j+} A^{i+} |u\rangle, f^+(f^+(E)) \} \quad \dots \\ \{ A^{i+} |u\rangle, f^+(E) \} \quad \nearrow \quad \{ A^{j-} A^{i+} |u\rangle, f^-(f^+(E)) \} \quad \dots \\ \{ A^{i-} |u\rangle, f^-(E) \} \quad \searrow \quad \{ A^{j+} A^{i-} |u\rangle, f^+(f^-(E)) \} \quad \dots \\ \{ A^{j-} A^{i-} |u\rangle, f^-(f^-(E)) \} \quad \dots \end{array} & \end{array} \quad (2.12)$$

As in the classical case ($q = 1$), starting from an arbitrary $\{|u\rangle, E\}$ we would generally get an unbounded (from below) energy spectrum $\{E, f^+(E), f^-(E), \dots\}$ (in

³In Ref. [5] the operators $a^{i\pm}$ were introduced in block-form as the collection $\{a_r^{i\pm}\}$ of their projections on the r -th eigenspace of h_ω (see eq. (2.29) below); the here presented more elegant form (where index r is replaced by the dependence on h_ω) was first introduced in Ref. [13].

the case $q \neq 1$ it would also be uncountable), which is non-physical. This is excluded in the case $q = 1$ by the condition that $|u\rangle$ is normalizable; this is equivalent to the condition that the energy spectrum generated starting from $\{|u\rangle, E\}$ is bounded from below. It is quite difficult to impose normalizability at this stage in the case $q \neq 1$, so we impose directly the second condition. This implies that one of the generated eigenvector is a “ground-state” $|0\rangle$:

$$A^{i-}|0\rangle = 0, \quad H_\omega|0\rangle = E_0|0\rangle, \quad (2.13)$$

where

$$E_0 = \omega(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2}]_q, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (2.14)$$

As a direct consequence, for the set of eigenvectors generated by the shower (2.8) applied to $|0\rangle$ it is true that $\{E = f^+(f^-(E)) = f^-(f^+(E))$ i.e. that $|u\rangle, A^{i-}A^{i+}|u\rangle, A^{j+}A^{j-}|u\rangle$ have the same energy (which was not true in general), so that the spectrum is countable (which is necessary for the separability of the Hilbert space) and A^{i+}, A^{i-} actually create and destroy an energy excitation respectively.

Let us call \mathcal{H} the linear span of all the vectors generated by the shower (2.8) starting from $|u\rangle = |0\rangle$. Let us now define

$$|i_r, i_{r-1}, \dots, i_1\rangle := A^{+i_r}A^{+i_{r-1}}\dots A^{+i_1}|0\rangle \quad (2.15)$$

$$\mathcal{H}_r := \text{Span}_{\mathbb{C}}\{|i_r, i_{r-1}, \dots, i_1\rangle, \quad i_j = -n, \dots, n\} \quad n \geq 0 \quad (2.16)$$

Proposition 2 $r \neq s$ implies $\mathcal{H}_r \perp \mathcal{H}_s$ w.r.t. the scalar product (2.4); \mathcal{H}_r has the same dimension $\binom{N+r-1}{N-1}$ as in the classical case and is an eigenspace of H_ω with eigenvalue E_r :

$$H_\omega \mathcal{H}_r = E_r \mathcal{H}_r \quad E_r = \omega(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2} + r]_q; \quad (2.17)$$

$A^{i\pm}(\mathcal{H}_r) = \mathcal{H}_{r\pm 1}$. $\mathcal{H} = \bigoplus_{r=1}^{\infty} \mathcal{H}_r$. Finally, \mathcal{H} endowed with the scalar product (2.4) is a pre-Hilbert space, and can be completed into a Hilbert space in the standard way.

Remark Note the $q \rightarrow q^{-1}$ invariance of the energy levels E_r . Both for $q > 1$ and for $0 < q < 1$ the difference $E_{r+1} - E_r$ increases and *diverges* with r , implying that it becomes a macroscopic energy gap for sufficiently large r (contrary to what happens with some other q -deformed harmonic oscillators [1]).

The definition of $A^{i\pm}$ we gave in equation (2.9) is not complete yet, since equation (2.10) fixes only the ratio $\frac{\beta^\pm(q, h\omega)}{\gamma^\pm(q, h\omega)}$, not β^\pm itself. One can choose the latter in such a way that annihilators/creators are hermitean conjugate of each other

$$(A^{i+})^\dagger = A^{i-}C_i \quad (A^{i-})^\dagger = A^{i+}C_i; \quad (2.18)$$

this condition still leaves some arbitrariness in the definition of β^\pm , amounting to the normalization of $A^{i+}A^{j-}$ (in Ref. [5] we removed it by the somewhat arbitrary condition that the operator $a^{i+} + a^{i-}$ didn't depend on ∂ ; any other choice would be legitimate).

Commutation relations of the following form hold for the creation/annihilation operators: $A^{i+}A^{j-}$ (B, l^{ij} were defined in eqs. (1.9), (1.10); the explicit rather lengthy form of the functions g_A, g_1, g'_1, g_S is not necessary here):

$$\mathcal{P}_A^{ij}{}_{hk} A^{h-} A^{k-} = 0 = \mathcal{P}_A^{ij}{}_{hk} A^{h+} A^{k+} \quad (2.19)$$

$$\mathcal{P}_A^{ij}{}_{hk} A^{h+} A^{k-} = g_A(h_\omega) l^{ij} \quad \mathcal{P}_A^{ij}{}_{hk} A^{h-} A^{k+} = -g_A(h_\omega) l^{ij} \quad (2.20)$$

$$A^{h+} A_h^- = g_1(h_\omega) B \quad A^{h-} A_h^+ = g'_1(h_\omega) B \quad (2.21)$$

$$\mathcal{P}_S^{ij}{}_{hk} A^{h+} A^{k-} - g_S(h_\omega) \mathcal{P}_S^{ij}{}_{hk} A^{h-} A^{k+} = 0 \quad (2.22)$$

Due to relation (2.18), $X^j := A^{j+} + A^{j-}$, $P^j := \frac{1}{i}(A^{j+} - A^{j-})$ have the same hermitean conjugation relations of x^i , therefore there exist N linearly independent combinations of the X^i 's (resp. P^i 's) which are hermitean operators. The latter can be called the position and momentum operators respectively, since in the limit $q \rightarrow 1$ they become the classical position and momentum operators respectively. They don't commute, rather “ q -commute”:

$$\mathcal{P}_A^{ij}{}_{hk} X^h X^k = 0. \quad \mathcal{P}_A^{ij}{}_{hk} P^h P^k = 0. \quad (2.23)$$

Since h_ω, \bar{h}_ω are scalars, $[l^{ij}, h_\omega] = 0 = [l^{ij}, \bar{h}_\omega]$. One can easily show that angular momentum operators coincide in the barred and unbarred scheme, so we can rewrite the above equations in an abstract form $[l^{ij}, H_\omega] = 0$ where now by l^{ij} we mean in fact (with a slight abuse of notation) the pair (l^{ij}, \bar{l}^{ij}) ; on these pairs hermitean conjugation coincides with complex conjugation. $l \cdot l$ is a scalar itself and therefore $[l \cdot l, l^{ij}] = 0$; moreover, it is hermitean, since it is real. Then we can diagonalize simultaneously $H_\omega, l \cdot l$ and a real Cartan subalgebra of $U_q(so(N))$. This is the q -deformed analogue of diagonalizing angular momentum observables and a hamiltonian with central potential in the classical case. When $N = 3, 4$ these operators will make up a complete set of commuting observables. The eigenvalues of $l \cdot l$ are

$$l_k^2 = [k]_q [k + N - 2]_q \frac{(q^{2-\frac{N}{2}} + q^{\frac{N}{2}-2})}{(q + q^{-1})(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1})} \quad k = 0, 1, 2, \dots \quad (2.24)$$

One can decompose each \mathcal{H}_r into the direct sum of eigenspaces of $l \cdot l$ corresponding to different eigenvalues l_k^2 . Summing up:

Proposition 3 $\mathcal{H}_{r,r-2m}$ ($r \geq 0, 0 \leq m \leq \frac{r}{2}$) is an eigenspace of the operators $H_\omega, l \cdot l$ with eigenvalues E_r, l_{r-2m}^2 (see (2.17), (2.24)) respectively. Moreover

$$\mathcal{H} = \bigoplus_{r=0}^{\infty} \bigoplus_{0 \leq m \leq \frac{r}{2}} \mathcal{H}_{r,r-2m}. \quad (2.25)$$

\oplus is to be understood in the sense of direct sum of mutually orthogonal subspaces w.r.t. the scalar product $\langle \cdot | \cdot \rangle$.

We give some explicit formulae regarding “wavefunctions”. The unbarred and barred solutions $\psi_0, \bar{\psi}_0 \in Fun(\mathbb{R}_q^N)$ of equation (2.13) are

$$\psi_0 := e_{q^2} \left[-\frac{q^{-N} \omega x C x}{1 + q^{2-N}} \right] \quad \bar{\psi}_0 := e_{q^{-2}} \left[-\frac{q^N \omega x C x}{1 + q^{N-2}} \right]; \quad (2.26)$$

according to our conventions $|0\rangle = (\psi_0, \bar{\psi}_0)$. Introducing the notation

$$|i_r, i_{r-1}, \dots, i_1\rangle =: (\psi_r^{i_r i_{r-1} \dots i_1}, \bar{\psi}_r^{i_r i_{r-1} \dots i_1}), \quad (2.27)$$

one finds

$$\psi_r^{i_r i_{r-1} \dots i_1} := a_r^{i_r+} a_{r-1}^{i_{r-1}+} \dots a_1^{i_1+} \psi_0 \quad \bar{\psi}_r^{i_r i_{r-1} \dots i_1} := \bar{a}_r^{i_r+} \bar{a}_{r-1}^{i_{r-1}+} \dots \bar{a}_1^{i_1+} \bar{\psi}_0, \quad (2.28)$$

where

$$a_h^{i+} := b_h(q) \left(x^i - \frac{q^{2-h}}{\omega} \partial^i \right) \Lambda^{-\frac{1}{2}} \quad \bar{a}_h^{i+} := b_h(q^{-1}) \left(x^i - \frac{q^{h-2}}{\omega} \bar{\partial}^i \right) \Lambda^{\frac{1}{2}} \quad i = 1, 2, \dots, N; \quad (2.29)$$

the pair $(a_h^{i+}, \bar{a}_h^{i+})$ is the explicit form of $A^{i+}|_{\mathcal{H}_{h-1}}$, the creation operator A^{i+} with domain of definition restricted to \mathcal{H}_{h-1} . Similarly, $A^{i-}|_{\mathcal{H}_{h-1}} =: (a_h^{i-}, \bar{a}_h^{i-})$ where

$$a_h^i := q^{2-2h-N} b_h(q) \left(x^i + \frac{q^{h+N}}{\omega} \partial^i \right) \Lambda^{-\frac{1}{2}} \quad \bar{a}_h^i := q^{-2+2h+N} b_h(q^{-1}) \left(x^i + \frac{q^{-h-N}}{\omega} \bar{\partial}^i \right) \Lambda^{\frac{1}{2}}; \quad (2.30)$$

b_h, d_h are given in Ref. [5, 7]. One can easily verify from equations (2.28) that

$$\begin{cases} \psi_r^{i_r i_{r-1} \dots i_1} = P_r(x) e_{q^2} \left[-\frac{q^{-N-r} \omega x C x}{(1+q^{2-N})} \right] \\ \bar{\psi}_r^{i_r i_{r-1} \dots i_1} = P'_r(x) e_{q^{-2}} \left[-\frac{q^{r+N} \omega x C x}{(1+q^{N-2})} \right] \end{cases} \quad (2.31)$$

where $P_r(x), P'_r(x)$ are two polynomials of degree r in x containing only terms of degree $r, r-2, r-4, \dots$, and $e_q[Z]$ denotes the q -exponential

$$e_q[Z] := \sum_{n=0}^{\infty} \frac{Z^n}{(n)_q!}, \quad (n)_q := \frac{q^n - 1}{q - 1}. \quad (2.32)$$

$\psi_r^{i_r i_{r-1} \dots i_1}, \bar{\psi}_r^{i_r i_{r-1} \dots i_1}$ can be called the unbarred, barred q -deformed Hermite functions, since they both reduce to the classical Hermite functions in the limit $q \rightarrow 1$.

Finally, the wavefunctions realizing the subspace $\mathcal{H}_{r,r-2m}$ ($r \geq 0, 0 \leq m \leq \frac{r}{2}$) are combinations of the $\psi_r^{i_r i_{r-1} \dots i_1}, \bar{\psi}_r^{i_r i_{r-1} \dots i_1}$ with fixed r . In the unbarred case they are given by

$$[(\mathcal{P}_1 \otimes' \dots \otimes' \mathcal{P}_1 \otimes' \mathcal{P}_{r-2m,S}) \psi_r]^{l_1 \dots l_r} \propto P_S^{l_{2m+1} \dots l_r}(x) p_{r,m}(xCx) e_{q^2} \left[-\frac{\omega q^{-r-N} x C x}{\mu} \right], \quad (2.33)$$

where $p_{r,m}(z)$, $0 \leq m \leq \frac{r}{2}$ are polynomials in one variable z and

$$P_S^{l_{2m+1} \dots l_n}(x) := \mathcal{P}_{r-2m, S, j_{2m+1} \dots j_r}^{l_{2m+1} \dots l_r} x^{j_{2m+1}} \dots x^{j_r} \quad (2.34)$$

are the “q-deformed spheric homogenous polynomials” of degree r obtained through application of the q-symmetric-modulo-trace projector $\mathcal{P}_{r-2m, S}$. The latter is defined by

$$\mathcal{P}_{k, S} \mathcal{P}_{Ai, (i+1)} = 0 = \mathcal{P}_{k, S} \mathcal{P}_{1i, (i+1)}, \quad \mathcal{P}_{Ai, (i+1)} \mathcal{P}_{k, S} = 0 = \mathcal{P}_{1i, (i+1)} \mathcal{P}_{k, S}, \quad (\mathcal{P}_{k, S})^2 = \mathcal{P}_{k, S}, \quad (2.35)$$

$1 \leq i \leq k-1$, where $\mathcal{P}_{Ai, (i+1)} = (\otimes \mathbf{1})^{i-1} \otimes \mathcal{P}_A \otimes (\otimes \mathbf{1})^{n-i-1}$, etc. The formulae in the barred case are similar. For further details on wavefunctions see also U. & S. Watamura in these proceedings.

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References

- [1] See for example: A. J. Macfarlane, J. Phys. A: Math. Gen. **22** (1989) 4581; L. C. Biedenharn, J. Phys. A: Math. Gen. **22** (1989) L873; Chang-Pu Sun and Hong-Chen Fu, J. Phys. A: Math. Gen. **22** (1989) L983.
- [2] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys. C Part. Fields **49** (1991) 439.
- [3] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, ” Quantization of Lie Groups and Lie Algebras ”, Algebra and Analysis, **1** (1989) 178, translated from the Russian in Leningrad Math. J. **1** (1990), 193.
- [4] G. Fiore, Int. J. Mod. Phys. **A7** (1992), 7597-7614.
- [5] G. Fiore, Int. J. Mod. Phys. **A8** (1993), 4679-4729.
- [6] G. Fiore, “Realization of $U_q(so(N))$ within the Differential Algebra on \mathbf{R}_q^N ”, Preprint SISSA 90/93/EP, hep-th9403033. To appear in Commun. Math. Phys.
- [7] G. Fiore, Ph.D. Thesis, SISSA.
- [8] G. Fiore, The Euclidean Hopf algebra $U_q(e^N)$ and its fundamental Hilbert space representations, Preprint Sissa 53/94/EP, LMU-TPW 94-11, hep-th 9407195.

- [9] S. Majid, J. Math. Phys. **34** (1993), 2045.
- [10] S. Majid, “ Hopf Algebras for Physics at the Planck Scale ”, J. Class. Q. Grav **5** (1988), 1587-1606.
- [11] O. Ogievetsky, Lett. Math. Phys. **24** (1992),245.
- [12] M. Schlieker, W. Weich and R. Weixler, Z. Phys. **C 53** (1992), 79-82;
- [13] U. Carow-Watamura and S. Watamura, Int. J. Mod. Phys.**A9** (1994), 3989-4008.
- [14] J. Wess, Talk given on occasion of the Third Centenary Celebrations of the Mathematische Gesellschaft Hamburg, March 1990; S. L. Woronowicz, Publ. Rims. Kyoto Univ. **23** (1987) 117.